The Vasicek Interest Rate Process
An Alternative Bond Price Equation

Gary Schurman, MBE, CFA

March, 2013

In this white paper we will develop the mathematics to price a bond using an alternative bond price equation. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are currently standing at time zero and are tasked with determining the price to be paid at time s for a bond that matures at time t given the expected short rate at time s. Our go-forward model assumptions are...

Description	Symbol	Value
Bond face value (in dollars)	F	1,000
Bond purchase time (in years)	s	3.00
Bond maturity time (in years)	t	7.00
Short rate at time zero	r_0	0.04
Long-term short rate mean	r_∞	0.09
Annualized short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

Question: What is the expected price to be paid for this bond at time s?

Distribution Of The Stochastic Short Rate

We defined the variable r_t to be the random short rate at time t. The short rate is defined as the annualized interest rate at which an entity can borrow money for an infinitesimally short period of time. Vasicek's stochastic differential equation that defines the change in the short rate r_t over the infinitesimally small time interval $[t, t + \delta t]$ is... [1]

$$\delta r_t = \lambda\,(r_\infty - r_t)\,\delta t + \sigma\,\delta W_t \text{ ...where... } \delta W_t \sim N\Bigg[0, \delta t\Bigg] \tag{1}$$

The equation for the mean of the short rate at time t given the short rate at time s is... [1]

$$\text{SR mean} = r_\infty + \text{Exp}\Bigg\{ -\lambda\,(t-s)\Bigg\}\Bigg(r_s - r_\infty\Bigg) \tag{2}$$

The equation for the variance of the short rate at time t given the short rate at time s is... [1]

$$\text{SR variance} = \frac{1}{2}\,\sigma^2\Bigg(1 - \text{Exp}\Bigg\{ -2\,\lambda\,(t-s)\Bigg\}\Bigg)\lambda^{-1} \tag{3}$$

Distribution Of The Stochastic Discount Rate

The equation for the random stochastic discount rate over the time interval $[s, t]$ is... [2]

$$\text{Random SDR} = \int\limits_s^t r_u\,\delta u \tag{4}$$

The equation for the mean of the stochastic discount rate at time t given the short rate at time s is... [2]

$$\text{SDR mean} = r_\infty\,(t-s) + (r_\infty - r_s)\left(\text{Exp}\left\{-\lambda\,(t-s)\right\} - 1\right)\lambda^{-1} \tag{5}$$

The equation for the variance of the stochastic discount rate at time t given the short rate at time s is... [2]

$$\text{SDR variance} = \frac{\sigma^2}{2\,\lambda^3}\left(2\,\lambda\,(t-s) - 3 + 4\,\text{Exp}\left\{-\lambda\,(t-s)\right\} - \text{Exp}\left\{-2\,\lambda\,(t-s)\right\}\right) \tag{6}$$

An Alternative Bond Price Equation

We defined the variable $P(s,t)$ to be the price at time s of a zero coupon bond that pays one dollar at time t given the known short rate at time s. The price of this bond may be written as the expectation of the path integral of the short rate over the time interval $[s,t]$. Using Equations (5) and (6) above the equation for bond price is... [3]

$$P(s,t) = \mathbb{E}\left[\text{Exp}\left\{-\int_s^t r_u\,\delta u\right\}\right] = \text{Exp}\left\{-\text{SDR mean} + \frac{1}{2}\,\text{SDR variance}\right\} \tag{7}$$

Using Equation (7) above the equation for the log of bond price is...

$$\ln P(s,t) = -\text{SDR mean} + \frac{1}{2}\,\text{SDR variance} \tag{8}$$

When pricing a forward bond the short rate at time s is not known at time zero and therefore the short rate at time s is a random variable. If the short rate is a random variable then the forward bond price is also a random variable with a given mean and variance. To determine the distribution of the forward bond price we want to isolate the random short rate r_s in the pricing equation and therefore we need an alternative bond pricing equation.

We will start by defining the function $B(s,t)$ as follows...

$$B(s,t) = \left(1 - \text{Exp}\left\{-\lambda\,(t-s)\right\}\right)\lambda^{-1} \tag{9}$$

Using Equation (9) above the equation for the square of the function $B(s,t)$ is...

$$B(s,t)^2 = \left(1 - 2\,\text{Exp}\left\{-\lambda\,(t-s)\right\} + \text{Exp}\left\{-2\,\lambda\,(t-s)\right\}\right)\lambda^{-2} \tag{10}$$

We will rewrite stochastic discount rate mean Equation (5) above as...

$$\text{SDR mean} = r_\infty\,(t-s) + (r_s - r_\infty)\left(1 - \text{Exp}\left\{-\lambda\,(t-s)\right\}\right)\lambda^{-1} \tag{11}$$

Using Equation (9) above we will rewrite Equation (11) above as...

$$\text{SDR mean} = r_\infty\,(t-s) + r_s\,B(s,t) - r_\infty\,B(s,t) \tag{12}$$

We will rewrite the negative of stochastic discount rate variance Equation (6) above as...

$$\begin{aligned}
-\text{SDR variance} &= \frac{\sigma^2}{2\,\lambda^3}\left(3 - 4\,\text{Exp}\left\{-\lambda\,(t-s)\right\} + \text{Exp}\left\{-2\,\lambda\,(t-s)\right\} - 2\,\lambda\,(t-s)\right) \\
&= \frac{\sigma^2}{2\,\lambda^3}\left(1 - 2\,\text{Exp}\left\{-\lambda\,(t-s)\right\} + \text{Exp}\left\{-2\,\lambda\,(t-s)\right\} + 2\left(1 - \text{Exp}\left\{-\lambda\,(t-s)\right\}\right) - 2\,\lambda\,(t-s)\right) \\
&= \frac{\sigma^2}{2\,\lambda}\,\frac{1}{\lambda^2}\left(1 - 2\,\text{Exp}\left\{-\lambda\,(t-s)\right\} + \text{Exp}\left\{-2\,\lambda\,(t-s)\right\} + 2\left(1 - \text{Exp}\left\{-\lambda\,(t-s)\right\}\right) - 2\,\lambda\,(t-s)\right)
\end{aligned} \tag{13}$$

Using Equations (9) and (10) above we will rewrite Equation (13) above as...

$$
\begin{aligned}
-\text{SDR variance} &= \frac{\sigma^2}{2\,\lambda}\left(B(s,t)^2 + \frac{2}{\lambda}\,B(s,t) - \frac{2}{\lambda}\,(t-s)\right)\\
-\text{SDR variance} &= \frac{\sigma^2}{\lambda^2}\left(B(s,t) - (t-s)\right) + \frac{\sigma^2}{2\,\lambda}\,B(s,t)^2\\
\text{SDR variance} &= -\frac{\sigma^2}{\lambda^2}\left(B(s,t) - (t-s)\right) - \frac{\sigma^2}{2\,\lambda}\,B(s,t)^2 \qquad (14)
\end{aligned}
$$

Note that we can rewrite stochastic discount rate mean and variance Equations (12) and (14) above as...

$$
\begin{aligned}
\text{SDR mean} &= r_\infty\,(t-s) + \left(r_s - r_\infty\right)B(s,t)\\
\text{SDR variance} &= \sigma^2\left[\left((t-s) - B(s,t)\right)\lambda^{-2} - \frac{1}{2}\,B(s,t)^2\,\lambda^{-1}\right] \qquad (15)
\end{aligned}
$$

Using Equations (12) and (14) above we can write bond pricing Equation (7) above as...

$$
\begin{aligned}
\ln P(s,t) &= -\text{SDR mean} + \frac{1}{2}\,\text{SDR variance}\\
&= -\left(r_\infty\,(t-s) + r_s\,B(s,t) - r_\infty\,B(s,t)\right) + \frac{1}{2}\left(-\frac{\sigma^2}{\lambda^2}\left(B(s,t) - (t-s)\right) - \frac{\sigma^2}{2\,\lambda}\,B(s,t)^2\right)\\
&= r_\infty\left(B(s,t) - (t-s)\right) - B(s,t)\,r_s - \frac{\sigma^2}{2\,\lambda^2}\left(B(s,t) - (t-s)\right) - \frac{\sigma^2}{4\,\lambda}\,B(s,t)^2\\
&= \left(r_\infty - \frac{\sigma^2}{2\,\lambda^2}\right)\left(B(s,t) - (t-s)\right) - \frac{\sigma^2}{4\,\lambda}\,B^2(s,t) - B(s,t)\,r_s \qquad (16)
\end{aligned}
$$

We will define the function $A(s,t)$ as follows...

$$
A(s,t) = \left(r_\infty - \frac{\sigma^2}{2\,\lambda^2}\right)\left(B(s,t) - (t-s)\right) - \frac{\sigma^2}{4\,\lambda}\,B^2(s,t) \qquad (17)
$$

Using Equation (17) above we can rewrite Equation (16) above as...

$$
\ln P(s,t) = A(s,t) - B(s,t)\,r_s \qquad (18)
$$

Using Equation (18) above the equation for bond price at time s is...

$$
P(s,t) = \text{Exp}\left\{A(s,t) - B(s,t)\,r_s\right\} \qquad (19)
$$

Answer To Our Hypothetical Problem

Question: What is the expected price to be paid for this bond at time s?

Using Equation (9) the equation for bond pricing parameter $B(s,t)$ is...

$$
B(3,7) = \left(1 - \text{Exp}\left\{-0.35 \times (7-3)\right\}\right) \times 0.35^{-1} = 2.1526 \qquad (20)
$$

Using Equations (17) and (20) the equation for bond pricing parameter $A(s,t)$ is...

$$
A(3,7) = \left(0.09 - \frac{0.03^2}{2 \times 0.35^2}\right) \times \left(2.1526 - (7-3)\right) - \frac{0.03^2}{4 \times 0.35} \times 2.1526^2 = -0.1625 \qquad (21)
$$

Using Equation (2) above the expected short rate at time s is...

$$
r_s = 0.09 + \text{Exp}\left\{-0.35 \times (3-0)\right\} \times \left(0.04 - 0.09\right) = 0.0725 \qquad (22)
$$

Using Equations (19) above and the bond price parameter estimates above the answer to the question is...

$$
P(3,7) = 1{,}000 \times \text{Exp}\left\{-0.1625 - 2.1526 \times 0.0725\right\} = 727.22 \qquad (23)
$$

References

[1] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Short Rate,* February, 2013.

[2] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Discount Rate,* February, 2013.

[3] Gary Schurman, *The Vasicek Interest Rate Process - Zero Coupon Bond Price,* February, 2013.